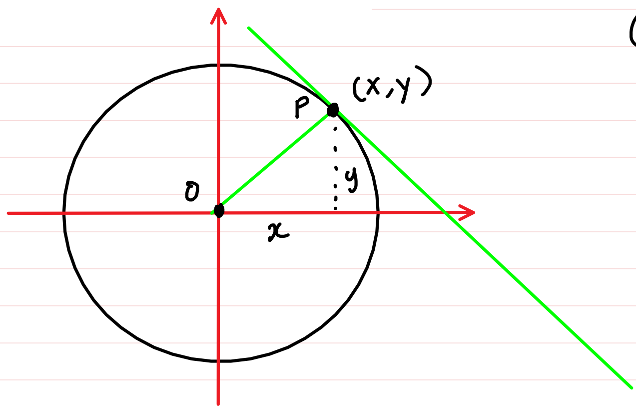
47. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.



Ingredients:

① 2 lines with slopes m_1 & m_2 are perpindicular if and only if
$m_1 m_2 = -1$

② A circle with centre $(0,0)$ radius r has equation:

diff^c implicitly $\longrightarrow$ $x^2 + y^2 = r^2$

m_2 = slope of tangent $= y'$

$\frac{d}{dx}$) $2x + 2yy' = 0$

$2yy' = -2x$

$m_2 \quad y' = \frac{-2x}{2y} = \frac{-x}{y}$

m_1 = Slope of OP $= \frac{\text{rise}}{\text{run}} = \frac{y}{x}$

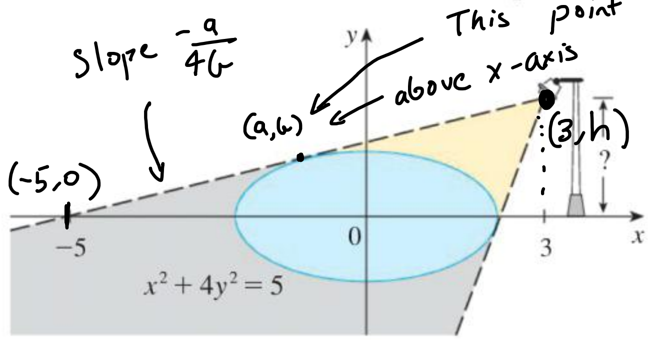
radius

$$m_1 \cdot m_2 = \left(\frac{y}{x}\right) \cdot \left(\frac{-x}{y}\right) = -1$$

✓✓

80. The figure shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 4y^2 \leqslant 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?

This point is on the ellipse!



(a,b) = point of tangency (don't call it (x,y)!)

big warning.

Slope of tangent is y' at (a,b)

Implicitly differentiate $x^2 + 4y^2 = 5$

$\frac{d}{dx}$) $2x + 8yy' = 0$

$y' = \frac{-2x}{8y} = \frac{-x}{4y}$

Slope of tangent @ (a,b) $= \frac{-a}{4b}$

As (a,b) is on the ellipse with eq^n $x^2 + 4y^2 = 5$

Equation of tangent:

$$\boxed{y - b = \frac{-a}{4b}(x - a)}$$

Sub in $x = -5 \quad y = 0$

replace $a = -1$ $b = 1$

$\times 4b$) $-b = -\frac{a}{4b}(-5 - a)$

$-4b^2 = 5a + a^2$

$0 = 5a + a^2 + 4b^2$

$-4b^2 = 5a + a^2$ $\qquad$ $x^2 + 4y^2 = 5$

$0 = 5a + \underbrace{a^2 + 4b^2} \qquad \underbrace{a^2 + 4b^2} = 5$

So $\qquad 0 = 5a + 5$

$-1 = a$ $\qquad$ As $a^2 + 4b^2 = 5$

plug in $a = -1$ $\qquad 1^2 + 4b^2 = 5$

$4b^2 = 4$

$b^2 = 1$

$b = +1$ or $\cancel{-1}$

$(a,b) = (-1, 1)$ $\qquad$ the tangent line

$y - 1 = \frac{1}{4}(x+1)$

plug in $(3, h)$ $\qquad$ h = height of light

$h - 1 = \frac{1}{4}(3+1)$

$h = 2.$