An important limit

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$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$ Remember (see text for derivation).

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = \lim_{\theta \to 0} \frac{(\cos\theta - 1)}{\theta} \cdot \frac{(\cos\theta + 1)}{(\cos\theta + 1)}$$

$$= \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)}$$

$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta - 1 = -\sin^2\theta$

$$= \lim_{\theta\to 0} \frac{-\sin^2\theta}{\theta(\cos\theta + 1)}$$

this has limit 1

Plug in!

$$= -\lim_{\theta\to 0} \left(\frac{\sin\theta}{\theta}\right) \cdot \left(\frac{\sin\theta}{\cos\theta + 1}\right) = -(1) \cdot \left(\frac{\sin 0}{\cos 0 + 1}\right) = \frac{(-1)(0)}{2}$$

$$= 0$$

Remember $$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0 \quad !$$

① $$\lim_{x\to 0} \frac{\sin 3x}{8x} = \lim_{x \to 0} \frac{\sin 3x}{3x \cdot \left(\frac{8}{3}\right)} = \left(\lim_{x\to 0} \frac{\sin 3x}{3x}\right) \cdot \frac{3}{8}$$

$c = 3$ $\quad \to 1$

$$= 1 \cdot \frac{3}{8}$$

Remark $c =$ number

$$\lim_{x\to 0} \frac{\sin cx}{cx} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$\theta = cx$

$\theta \to 0$ as $x \to 0$

② $$\lim_{x\to 0} \frac{\sin x}{\sin \pi x} = \lim_{x \to 0} \frac{x}{x} \cdot \frac{\sin x}{\sin \pi x}$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right) \cdot \left(\frac{x}{\sin \pi x}\right)$$

Use $$\lim_{x\to 0} \frac{\sin cx}{cx} = 1$$ again

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{\pi x}{\sin \pi x}\right) \cdot \frac{1}{\pi}$$

π's cancel to preserve equality

$$= \lim_{x\to 0} \underbrace{\cancel{\left(\frac{\sin x}{x}\right)}}_{} ^{1} \cdot \frac{1}{\cancel{\left(\frac{\sin \pi x}{\pi x}\right)}_{1}} \cdot \frac{1}{\pi}$$

$$= \lim_{x\to 0} \left(\frac{\cancel{x}}{\cancel{x}}\right) \cdot \left(\frac{\cancel{\sin \pi x}}{\cancel{\pi x}}\right) \quad \pi$$

$$= 1 \cdot \frac{1}{1} \cdot \frac{1}{\pi} = \frac{1}{\pi}$$

(3) $$\lim_{x\to 0} \frac{\sin x^2}{x} = \lim_{u\to 0^+} \frac{\sin u}{\pm\sqrt{u}}$$

$$= \pm \lim_{u\to 0^+} \frac{\sin u}{u^{1/2}}$$

$u = x^2 \quad \pm\sqrt{u} = x$

as $x \to 0$

$u \to 0^+$

$$= \pm \lim_{u\to 0^+} \left(\frac{\sin u}{u}\right) \cdot \left(u^{1/2}\right)$$

As $\frac{u^{\frac{1}{2}}}{u} = \frac{1}{u^{1/2}}$ by exponent laws

$$= \pm 1 \cdot \sqrt{0} = 0$$

(4) $$\lim_{x\to 0} \frac{\sin 3x \quad \sin 5x}{x^2} = \lim_{x\to 0} \frac{\sin 3x}{x} \cdot \frac{\sin 5x}{x}$$

$$= \lim_{x\to 0} 3 \cdot \left(\frac{\sin 3x}{3x}\right) \cdot 5\left(\frac{\sin 5x}{5x}\right)$$

Recall c = number

$$\lim_{x\to 0} \frac{\sin cx}{cx} = 1$$

$$= 15 \lim_{x\to 0} \left(\frac{\sin 3x}{3x}\right)\left(\frac{\sin 5x}{5x}\right) = 15$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta + \tan\theta} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta + \frac{\sin\theta}{\cos\theta}}$$

$$= \lim_{\theta\to 0} \frac{\sin\theta}{\left(\frac{\theta\cos\theta + \sin\theta}{\cos\theta}\right)}$$

$$= \lim_{\theta\to 0} \frac{\sin\theta\cos\theta}{\theta\cos\theta + \sin\theta} = L = 2$$

Trick

compute $\frac{1}{L}$

$$\frac{1}{L} = \lim_{\theta\to 0} \frac{\theta\cos\theta + \sin\theta}{\sin\theta\cos\theta}$$

$$= \lim_{\theta\to 0} \frac{\theta \cancel{\cos\theta}}{\sin\theta \cancel{\cos\theta}} + \frac{\cancel{\sin\theta}}{\cancel{\sin\theta}\cos\theta}$$

plug in

$$= \lim_{\theta\to 0} \left(\frac{\theta}{\sin\theta}\right) + \left(\frac{1}{\cos\theta}\right)$$

reciprocal of special limit

$$= \frac{1}{1} + \frac{1}{\cos 0} = 2 = \frac{1}{L}$$