Derivatives and rates of change

Saturday, March 7, 2015 12:17 PM

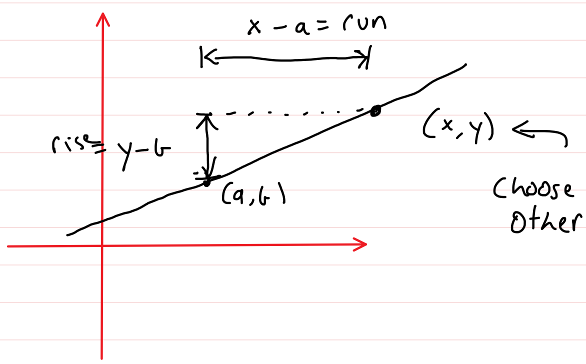
Equations of lines (Review)

** Don't memorize, understand **

slope $= \dfrac{\text{rise}}{\text{run}}$.

- Equation of line with given slope and point

slope $= m$ point $= (a,b)$



Choose some other point

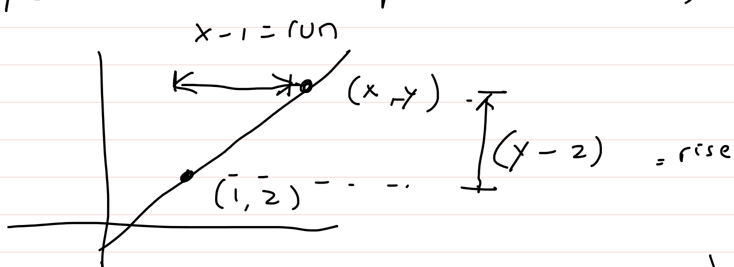
$$m = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y-b}{x-a}$$

$$m = \frac{y-b}{x-a}$$

$$m(x-a) = y-b \quad ✓$$

In action:

<u>Ex</u> slope $= 2$ point $= (1,2)$



$$2 = \text{slope} = \frac{y-2}{x-1}$$

$$2(x-1) = y-2$$

$$2x - 2 = y - 2$$

$$2x = y \quad ✓$$

Nothing to memorize!

Equation of line through 2 given points

E.g line through $\underset{\text{first}}{(2,1)}$ & $\underset{\text{second}}{(3,0)}$

$$\text{slope} = \frac{1-0}{2-3} = \frac{1}{-1} = -1$$

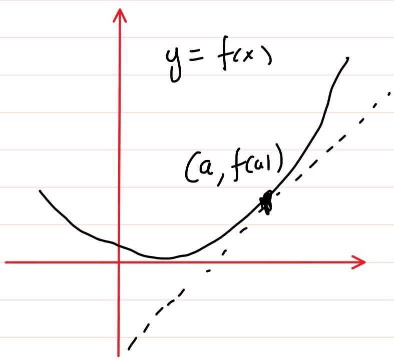
Now use above to find eq^n choose ~~other~~ one more point

(x, y) . $(2, 1) \longleftarrow$ Use this

$$-1 = \text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{y-1}{x-2}$$

Nothing to Memorize!

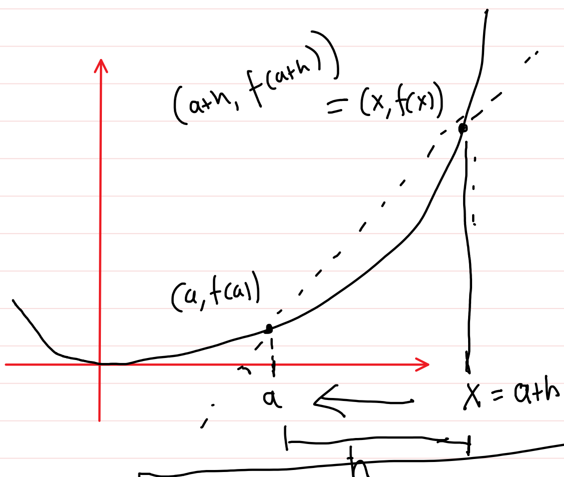
$$-1\,(x-2) = y-1$$
$$-x+2 = y-1$$
$$-x+3 = y$$



Tangent line at $(a, f(a))$ is the line that just touches the graph at $(a, f(a))$

$f'(a)$ = derivative of $f(x)$ at a

= slope of tangent at $(a, f(a))$



To compute $f'(a)$ choose another point on graph : $(x, f(x))$

The line through these 2 points has slope

$$\frac{f(x) - f(a)}{x - a} = m \quad \text{slope}$$

As $x \to a$ the dotted line turns into the tangent

So //

$$\boxed{f'(a) = \text{slope of tangent} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

remember

Example

$f(t) = 2t^2 + t$ Find $f'(1)$

$f(1) = 3$

$$f'(1) = \lim_{t \to 1} \frac{f(t) - f(1)}{t - 1}$$

$$= \lim \frac{2t^2 + t - 3}{}$$

$$= \lim_{t \to 1} \frac{2t^2 + t - 3}{t - 1}$$

$$= \lim_{t \to 1} \frac{(2t+3)\cancel{(t-1)}}{\cancel{(t-1)}}$$

$$= \lim_{t \to 1} 2t + 3 = 5 \checkmark$$

The derivative as a rate of change

Example: A particle moves so that its distance from the origin after t seconds is

$$s(t) = t^2 + 1$$

What is its velocity when $t = 1$ s.

velocity = rate of change of s (with respect to time t)

We want velocity at an instant.

change in distance

average velocity over interval $[1,2]$ $= \dfrac{s(2) - s(1)}{2-1} = \dfrac{5-2}{2-1} = 3$

chane in time

" " " " $[1, 1.5] = \dfrac{s(1.5) - s(1)}{1.5 - 1} = 2.5$

" " " " $[1, 1.25] = \dfrac{s(1.25) - s(1)}{1.25 - 1} = 2.25$

" " " " $[1, 1.1] = \dfrac{s(1.1) - s(1)}{1.1 - 1} = 2.1$

Better approx.

Answer $= v(1) = \lim_{t \to 1} \dfrac{s(t) - s(1)}{t - 1} = s'(1)$ derivative!

$$= \lim_{t \to 1} \frac{t^2 + 1 - 2}{t - 1}$$

$$= \lim_{t \to 1} \frac{t^2 - 1}{t - 1}$$

$$= \lim_{t \to 1} \frac{(t-1)(t+1)}{t - 1}$$

$$= \lim_{t \to 1} t+1 = 2$$

Remember: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Example ① $\lim_{x \to \frac{1}{4}} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}} = f'(a)$ what is $f(x)$ & a?

Exact match

Guess $f(x) = \frac{1}{x}$, $a = \frac{1}{4}$: Check: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

correct answer.

$$= \lim_{x \to \frac{1}{4}} \frac{\frac{1}{x} - 1/(1/4)}{x - \frac{1}{4}} = \lim_{x \to \frac{1}{4}} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}}$$

② $f'(a) = \lim_{h \to 0} \frac{e^{-2+h} - e^{-2}}{h}$ Guess $f(x) = e^{-x}$, $a = 2$

Check $f'(a) = \lim_{\lambda \to 0} \frac{f(a+\lambda) - f(a)}{\lambda} = \lim_{\lambda \to 0} \frac{e^{-(2+\lambda)} - e^{-2}}{\lambda}$ Not quite!

$$= \lim_{\lambda \to 0} \frac{e^{-2-\lambda} - e^{-2}}{\lambda}$$

use different variable, you'll see why

Change var: $-\lambda = h$

$\lambda = -h$

$\lambda \to 0 \quad h \to 0$

$$= \lim_{h \to 0} \frac{e^{-2+h} - e^{-2}}{-h}$$

$$= -\left(\lim_{h \to 0} \frac{e^{-2+h} - e^{-2}}{h} \right)$$

negative of correct thing.

correct answer $f(x) = -e^{-x}$ $a = 2$

fixes sign.

Another correct answer $f(x) = e^{x}$ $a = -2$.

Example $f'(2) = ?$ when $f(x) = \sqrt{x}$

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$\lim_{x \to 2} \frac{\quad}{x - 2}$$

$$= \lim_{x \to 2} \frac{\left(\sqrt{x} - \sqrt{2}\right)}{x - 2} \cdot \frac{\left(\sqrt{x} + \sqrt{2}\right)}{\sqrt{x} + \sqrt{2}}$$

multiply by conjugate

$$= \lim_{x \to 2} \frac{\sqrt{x}^{2} - \sqrt{2}^{2}}{(x - 2)\left(\sqrt{x} + \sqrt{2}\right)}$$

diff of 2 squares

$$= \lim_{x \to 2} \frac{\cancel{(x - 2)}}{\cancel{(x - 2)}\left(\sqrt{x} + \sqrt{2}\right)} = \lim_{x \to 2} \frac{1}{\sqrt{x} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$$