Limits at infinity : Part I

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1 Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

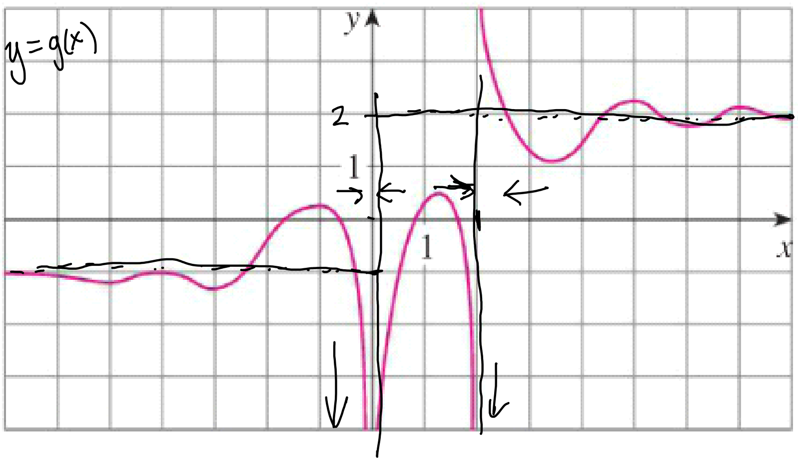
means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

9 Definition of an Infinite Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \quad \text{then} \quad f(x) > M$$



4. For the function g whose graph is given, state the following.

(a) $\lim_{x \to \infty} g(x)$ (b) $\lim_{x \to -\infty} g(x)$

(c) $\lim_{x \to 0} g(x)$ (d) $\lim_{x \to 2^-} g(x)$

(e) $\lim_{x \to 2^+} g(x)$ (f) The equations of the asymptotes

a) $\lim_{x \to \infty} g(x) = 2$

b) $\lim_{x \to -\infty} g(x) = -1$

c) $\lim_{x \to 0} g(x) = -\infty$

d) $\lim_{x \to 2^-} g(x) = -\infty$

e) $\lim_{x \to 2^+} g(x) = +\infty$

(f) Horizontal : $y = 2$, $y = -1$

Vertical : $x = 0$ $x = 2$.

Horizontal asymptotes lines $y = L$ with $L = \lim_{x \to \infty} g(x)$ or $\lim_{x \to -\infty} g(x) = L$

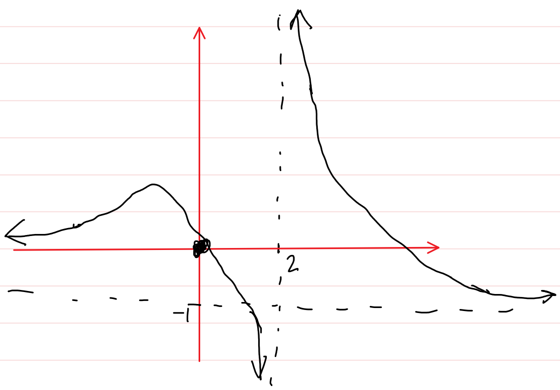
Vertical asymptotes lines $x = a$ with $\lim_{x \to a} g(x) = \pm\infty$

or 1-sided limit

Example Sketch a function $y = g(x)$ that satisfies

$\lim_{x \to 2^+} g(x) = +\infty$ $\lim_{x \to 2^-} g(x) = -\infty$ $g(0) = 0$

$\lim_{x \to -\infty} g(x) = 0$ $\lim_{x \to \infty} g(x) = -1$ (Many answers)



The limit laws apply to limits at ∞: with the exception of

$\lim_{x \to \infty} x^n \neq \infty^n$ ← what does this even mean?

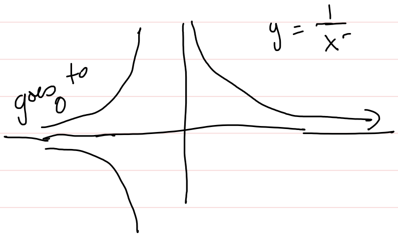
$\lim_{x \to \infty} \sqrt[n]{x} = \sqrt[n]{\infty}$

5 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$



e.g. lim $\frac{1}{}$ doesn't make eg $\lim_{x \to -\infty} \frac{1}{\sqrt{x}}$ doesn't make sense

as $\sqrt{\text{negative}}$ = not real!

Example $\lim_{x \to \infty} \frac{3x^3 + 5x + 1}{7x^3 + 7x^2 + 1} =$

Find highest power of x in denominator : x^3
divide everything by x^3

$$= \lim_{x \to \infty} \frac{\frac{3x^3 + 5x + 1}{x^3}}{\frac{7x^3 + 7x^2 + 1}{x^3}}$$

$$= \lim_{x \to \infty} \frac{\frac{3\cancel{x^3}}{\cancel{x^3}} + \frac{5\cancel{x}}{x^{\cancel{3}2}} + \frac{1}{x^3}}{\frac{7\cancel{x^3}}{\cancel{x^3}} + \frac{7\cancel{x^2}}{x^{\cancel{3}}} + \frac{1}{x^3}}$$

$$= \lim_{x \to \infty} \frac{3 + \frac{5}{x} + \frac{1}{x^3}}{7 + \frac{7}{x} + \frac{1}{x^3}}$$

Now apply theorem

$$= \frac{3 + 0 + 0}{7 + 0 + 0} = \frac{3}{7}$$

Example $\lim_{x \to \infty} \frac{x^2}{x+1} = \lim_{x \to \infty} \frac{\frac{x^2}{x}}{\frac{x}{x} + \frac{1}{x}}$ (Skip a little algebra)

same procedure
highest power of x in denominator is x

$$= \lim_{x \to \infty} \frac{x}{1 + \frac{1}{x}}$$

when x is large $\frac{\text{big}}{1 + \text{small}}$ = very big

$$= +\infty.$$

Example $\lim_{x \to \infty} \frac{2x+1}{\sqrt{x^2+1}} = \lim_{x \to \infty} \frac{\frac{2x}{x} + \frac{1}{x}}{\frac{\sqrt{x^2+1}}{x}}$ (Move x inside $\sqrt{\ }$ via : $x = \sqrt{x^2}$)

this time highest power is $\sqrt{x^2} = x$.

$$= \lim_{x \to \infty} \frac{2 + \frac{1}{x}}{\frac{\sqrt{x^2+1}}{\sqrt{x^2}}}$$

$$= \lim_{x \to \infty} \frac{2 + \frac{1}{x}}{\sqrt{\frac{x^2+1}{x^2}}}$$

1. $2 + \frac{1}{x}$

$$= \lim_{x \to \infty} \frac{2 + \frac{1}{x}}{\sqrt{\frac{x^2}{x^2} + \frac{1}{x^2}}}$$

$$= \lim_{x \to \infty} \frac{2 + \frac{1}{x}}{\sqrt{1 + \frac{1}{x^2}}} = \frac{2 + 0}{\sqrt{1 + 0}} = 2$$