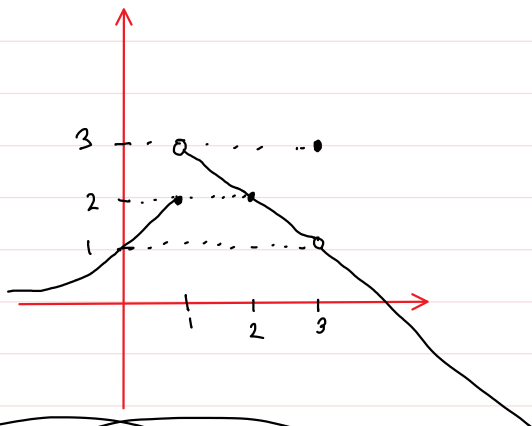
# Continuity

Monday, March 2, 2015 9:02 AM

A function is said to be *continuous* at a number *a* if
$$\lim_{x\to a} f(x) = f(a)$$




This function is in fact cts at all a except $a=1,3$.

continuous

Example Here is the graph of a function. Is the function continuous at $a = 1, 2, 3$?

$a=1$ : $\lim_{x\to 1^-} f(x) = 2$ $\lim_{x\to 1^+} f(x) = 3$

So $\lim_{x\to 1} f(x) =$ DNE, not cts

continuous

$a=2$ $\lim_{x\to 2} f(x) = 2 = f(2)$

By definition is cts at $a=2$

$a=3$ $\lim_{x\to 3} f(x) = 1$ But $f(3) = 3$

$\lim_{x\to 3} f(x) \neq f(3)$ Not cts

Example

$$f(x) = \begin{cases} \dfrac{1}{x+2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$$

For what $a$ is this function cts?

If $a \neq -2$ then $\lim_{x\to a} f(x) = \lim_{x\to a} \frac{1}{x+2} = \frac{1}{a+2} = f(a)$

plug in,

via limit laws

If $a = -2$ then $f(a) = 1$. $\lim_{x \to -2^+} \frac{1}{x+2} = +\infty$ $\left( \frac{\text{positive}}{\text{small positive}} \right)$

So $\lim_{x \to -2} f(x) = \text{DNE}$, $\lim_{x \to -2^-} \frac{1}{x+2} = -\infty$ $\left( \frac{\text{positive}}{\text{small negative}} \right)$

hence not cts at $a = -2$.

**4 Theorem** If $f$ and $g$ are continuous at $a$ and $c$ is a constant, then the following functions are also continuous at $a$:

1. $f + g$
2. $f - g$
3. $cf$
4. $fg$
5. $\frac{f}{g}$ if $g(a) \neq 0$

Example Suppose $f(x)$ & $g(x)$ are cts at $a = 2$ and $f(2) = 3$, $g(2) = \pi$.

What is $\lim_{x \to 2} 3f(x) + 5g(x)$ ?

By the theorem $3f(x) + 5g(x)$ is cts at $a = 2$:

So

$$\lim_{x \to 2} 3f(x) + 5g(x) = 3f(2) + 5g(2)$$
$$= 3(3) + 5\pi = 9 + 5\pi$$

**7 Theorem** The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

Example Show that $f(x) = \frac{\sin x}{\sqrt{1+x}}$ is cts in its domain.

$\left. \begin{array}{l} g(x) = \sin x \\ h(x) = \sqrt{1+x} \end{array} \right\}$ continuous in their domains by theorem.

$0 = h(x)$ when $x = -1$

Hence $f(x) = \dfrac{g(x)}{h(x)}$ is cts in its domain except when $x = -1$

However $x = -1$ is not in the domain of $f(x)$ <u>Conclusion</u>: $f(x)$ is cts in its domain.

<u>Example</u> Show that $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$

is cts.

$f(x)$ is cts at all $x$ except possibly $x = \pi/4$ as sine & cosine are cts by the theorem.

at $x = \pi/4$:

$$\lim_{x \to \pi/4^+} f(x) = \lim_{x \to \pi/4} \cos x = \cos \pi/4 = \frac{1}{\sqrt{2}}$$

$$\lim_{x \to \pi/4^-} f(x) = \lim_{x \to \pi/4^-} \sin x = \sin \pi/4 = \frac{1}{\sqrt{2}}$$

So $\displaystyle\lim_{x \to \pi/4} f(x) = \frac{1}{\sqrt{2}}$ & $f(\pi/4) = \cos \pi/4 = \dfrac{1}{\sqrt{2}}$.

As $\displaystyle\lim_{x \to \pi/4} f(x) = f(\pi/4)$ $f(x)$ is cts.

<u>Example</u> For which $x$ is the function

$$f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \leq x \leq 1 \\ 1/x & \text{if } x \geq 1 \end{cases}$$ cts?

It cts at all $x$ except possibly $x = -1, 1$ as by the theorem $x^2$, $x$ & $\frac{1}{x}$ ← when $x \geq 1$ are cts.

<u>Lets check $x = -1$</u>

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1} x^2 = 1$$

$\Big\}$ $\lim f(x) = DNE$

Lets check $x=-1$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1} x^2 = 1$$
$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x = -1$$
$$\lim_{x\to -1} f(x) = DNE$$

So $f(x)$ is not cts at $x=-1$

Lets check $x=1$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} \frac{1}{x} = 1$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$$
$$\lim_{x\to 1} f(x) = 1$$

$f(1) = \frac{1}{1} = 1$ As $f(1) = \lim_{x\to 1} f(x)$, $f(x)$ is cts at $x=1$.

Example Show that

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \end{cases}$$

is cts.

The function is cts at all $x$ except $x=0$

why: $\sin x$ is cts & $\frac{1}{x}$ is cts $(x \neq 0)$

a composition of cts functions is cts

So $\sin\left(\frac{1}{x}\right)$ is cts except when $x=0$

A product of cts functions is cts hence

$x^2 \sin\left(\frac{1}{x}\right)$ is cts.

This takes care of continuity at all $x$ except $x=0$

At $x=0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

Apply squeeze thm

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$
$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

$f(0) = 0 = \lim_{x\to 0} f(x)$

So $f(x)$ is cts at $x=0$.

$$-x^2 \le x^2 \sin\left(\tfrac{1}{x}\right) \le x^2 \quad \text{at} \quad x = 0.$$

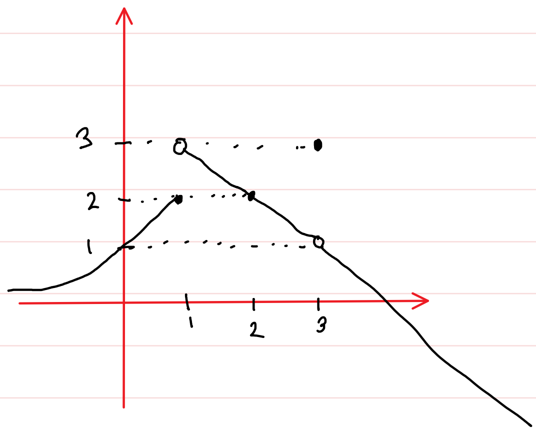
$$\underset{0}{\downarrow} \qquad\qquad \underset{0}{\downarrow}$$

**2 Definition** A function $f$ is **continuous from the right at a number $a$** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and $f$ is **continuous from the left at $a$** if

$$\lim_{x \to a^-} f(x) = f(a)$$




$$\lim_{x \to 1^-} f(x) = 2 = f(1)$$

so $f(x)$ is left cts at 1

$$\lim_{x \to 1^+} f(x) = 3 \ne 2 = f(1)$$

Not right cts at 1.